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Abstract. We describe all weighted Radon transforms on the plane for which the Chang approximate inversion formula is precise. Some subsequent results, including the Cormack type inversion for these transforms, are also given.

1.Introduction

We consider the weighted ray transformation P_W defined by the formula

$$\begin{aligned}&P_W f(s,\theta) = \int\limits_{\mathbb{R}} W(s\theta^\perp + t\theta, \theta) f(s\theta^\perp + t\theta) dt,\\ &s \in \mathbb{R},\ \theta = (\theta_1, \theta_2) \in \mathbb{S}^1,\ \theta^\perp = (-\theta_2, \theta_1),\end{aligned} \tag{1}$$

where $W = W(x,\theta)$ is the weight, $f = f(x)$ is a test function, $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$. Up to change of variables, P_W is known also as weighted Radon transform on the plane.

We recall that in definition (1) the product $\mathbb{R}\times\mathbb{S}^1$ is interpreted as the set of all oriented straight lines in $\mathbb{R}^2$. If $\gamma = (s,\theta) \in \mathbb{R}\times\mathbb{S}^1$, then $\gamma = \{x \in \mathbb{R}^2 :\ x = s\theta^\perp + t\theta,\ t \in \mathbb{R}\}$ (modulo orientation) and θ gives the orientation of γ.

We assume that

$$\begin{aligned}&W \ \text{ is complex} - \text{valued},\\ &W \in C(\mathbb{R}^2 \times \mathbb{S}^1) \cap L^\infty(\mathbb{R}^2 \cap \mathbb{S}^1),\\ &w_0(x) \overset{\text{def}}{=} \frac{1}{2\pi}\int\limits_{\mathbb{S}^1} W(x,\theta) d\theta \neq 0, \quad x \in \mathbb{R}^2,\end{aligned} \tag{2}$$

where $d\theta$ is the standard element of arc length on $\mathbb{S}^1$.

If $W \equiv 1$, then P_W is known as the classical ray (or Radon) transform on the plane. If

$$\begin{aligned}&W(x,\theta) = \exp(-Da(x,\theta)),\\ &Da(x,\theta) = \int\limits_0^{+\infty} a(x+t\theta) dt,\end{aligned} \tag{3}$$

where a is a complex-valued sufficiently regular function on $\mathbb{R}^2$ with sufficient decay at infinity, then P_W is known as the attenuated ray (or Radon) transform.

The classical Radon transform arises, in particular, in the X-ray transmission tomography. The attenuated Radon transform (at least, with $a \geq 0$) arises, in particular, in the single photon emission computed tomography (SPECT). Some other weights W also arise in applications. For more information in this connection see, for example, [Na], [K].

Precise and simultaneously explicite inversion formulas for the classical and attenuated Radon transforms were given for the first time in [R] and [No], respectively. For some other weights W precise and simultaneously explicite inversion formulas were given in [BS], [G].

On the other hand, the following Chang approximate inversion formula for P_W, where W is given by (3) with $a \geq 0$, is used for a long time, see [Ch], [M], [K]:

$$\begin{aligned} f_{appr}(x) &= \frac{1}{4\pi w_0(x)} \int\limits_{\mathbb{S}^1} h'(x\theta^{\perp}, \theta)d\theta, \quad h'(s,\theta) = \frac{d}{ds}h(s,\theta), \\ h(s,\theta) &= \frac{1}{\pi} p.v. \int\limits_{\mathbb{R}} \frac{P_W f(t,\theta)}{s-t} dt, \quad s \in \mathbb{R}, \ \theta \in \mathbb{S}^1, \ x \in \mathbb{R}^2, \end{aligned} \tag{4}$$

where w_0 is defined in (2). It is known that (4) is efficient as the first approximation in SPECT reconstructions and, in particular, is sufficiently stable to the strong Poisson noise of SPECT data. The results of the present note consist of the following:

(1) In Theorem, under assumptions (2), we describe all weights W for which the Chang approximate inversion formula (4) is precise, that is $f_{appr} \equiv f$ on $\mathbb{R}^2$;

(2) For P_W with W of Theorem we give also the Cormack type inversion (see Remark A) and inversion from limited angle data (see Remark B).

These results are presented in detail in the next section. In addition, we give also an explanation of efficiency of the Chang formula (4) as the first approximation in SPECT reconstructions (on the level of integral geometry).

2. Results

Let

$$\begin{aligned} &C_0(\mathbb{R}^2) \ \text{ denote the space of continuous} \\ &\text{compactly supported functions on } \ \mathbb{R}^2. \end{aligned} \tag{5}$$

Let

$$\begin{aligned} &L^{\infty,\sigma}(\mathbb{R}^2) = \{f : \ M^{\sigma} f \in L^{\infty}(\mathbb{R}^2)\}, \\ &M^{\sigma} f(x) = (1+|x|)^{\sigma} f(x), \quad x \in \mathbb{R}^2, \quad \sigma \geq 0. \end{aligned} \tag{6}$$

Theorem. *Let assumptions (2) hold and let $f_{appr}(x)$ be given by (4). Then*

$$f_{appr} = f \ \ (\textit{in the sense of distributios}) \ \textit{on} \ \ \mathbb{R}^2 \ \ \textit{for all} \ \ f \in C_0(\mathbb{R}^2) \tag{7}$$

if and only if

$$W(x,\theta) - w_0(x) \equiv w_0(x) - W(x,-\theta), \quad x \in \mathbb{R}^2, \quad \theta \in \mathbb{S}^1. \tag{8}$$

(This result remains valid with $C_0(\mathbb{R}^2)$ replaced by $L^{\infty,\sigma}(\mathbb{R}^2)$ for $\sigma > 1$.)

Theorem 1 is based on the following facts:

- Formula (4) coincides with the classical Radon inversion formula if $W \equiv 1$ and, as a corollary, is precise if $W \equiv w_0$.
- Formula (4) is equivalent to the symmetrized formula

$$\begin{aligned} f_{appr}(x) &= \frac{1}{4\pi w_0(x)} \int\limits_{\mathbb{S}^1} g'(x\theta^\perp, \theta) d\theta, \ x \in \mathbb{R}^2, \\ g(s,\theta) &= \frac{1}{2\pi} p.v. \int\limits_{\mathbb{R}} \frac{P_W f(t,\theta) + P_W f(-t,-\theta)}{s-t} dt, \quad (s,\theta) \in \mathbb{R} \times \mathbb{S}^1. \end{aligned} \tag{9}$$

- The following formula holds:

$$\begin{aligned} &\frac{1}{2}(P_W f(s,\theta) + P_W f(-s,-\theta)) = P_{W_{sym}} f(s,\theta), (s,\theta) \in \mathbb{R} \times \mathbb{S}^1, \\ &W_{sym}(x,\theta) = \frac{1}{2}(W(x,\theta) + W(x,-\theta)), \quad x \in \mathbb{R}^2, \ \theta \in \mathbb{S}^1. \end{aligned} \tag{10}$$

- If

$$\begin{aligned} &q \in C(\mathbb{R} \times \mathbb{S}^1), \ supp\, q \ \text{ is compact}, \ \ q(s,\theta) = q(-s,-\theta), \\ &g(s,\theta) = \frac{1}{\pi} p.v. \int\limits_{\mathbb{R}} \frac{q(t,\theta)}{s-t} dt, \quad (s,\theta) \in \mathbb{R} \times \mathbb{S}^1, \\ &\int\limits_{\mathbb{S}^1} g'(x\theta^\perp, \theta) d\theta \equiv 0 \ \text{ as a distribution of } \ x \in \mathbb{R}^2, \end{aligned} \tag{11}$$

  then $q \equiv 0$ on $\mathbb{R} \times \mathbb{S}^1$.

The statement that, under the assumptions of Theorem, property (8) implies (7) can be also deduced from considerations developed in [K].

Using that $W_{sym} \equiv w_0$ under condition (8), we obtain also the following

Remarks. Let conditions (2), (8) be fulfiled. Let $f \in C_0(\mathbb{R}^2)$. Then:

(A) $P_W f$ on $\Omega(D)$ uniquely determines f (or more precisely $w_0 f$) on $\mathbb{R}^2 \backslash D$ via (10) and the Cornack inversion from $P_{W_{sym}} f$ on $\Omega(D)$, where D is a compact in $\mathbb{R}^2$, $\Omega(D)$ denotes the set of all straight lines in $\mathbb{R}^2$ which do not intersect D;

(B) $P_W f$ on $\mathbb{R} \times (S_+ \cup S_-)$ uniquely determines f on $\mathbb{R}^2$ via (10) and standard inversion from the limited angle data $P_{W_{sym}} f$ on $\mathbb{R} \times S_+$, where S_+ is an arbitrary nonempty open connected subset of $\mathbb{S}^1$, $S_- = -S_+$.

For the case when W is given by (3) under the additional conditions that $a \geq 0$ and $supp\, a \subset D$, where D is some known bounded domain which is not too big, and for $f \in C(\mathbb{R}^2)$, $f \geq 0$, $supp\, f \subset D$, the transform $P_W f$ is relatively well approximated by $P_{W_{appr}} f$, where $W_{appr}(x,\theta) = w_0(x) + (1/2)(W(x,\theta) - W(x,-\theta))$. In addition, this W_{appr} already satisfies (8). This explains the efficiency of (4) as the first aproximation in SPECT reconstructions (on the level of integral geometry).

References

[BS] J.Boman and J.O.Strömberg, Novikov's inversion formula for the attenuated Radon transform - a new approach, *J.Geom.Anal.* **14** (2004), 185-198

[Ch] L.T.Chang, A method for attenuation correction in radionuclide computed tomography, *IEEE Trans. Nucl. Sci.* NS-25 (1978), 638-643

[G] S.Gindikin, A remark on the weighted Radon transform on the plane, *Inverse Problems and Imaging* **4** (2010), 649-653

[K] L.A.Kunyansky, Generalized and attenuated Radon transforms: restorative approach to the numerical inversion, *Inverse Problems* **8** (1992), 809-819

[M] K.Murase, H.Itoh, H.Mogami, M.Ishine, M.Kawamura, A.Lio and K.Hamamoto, A comparative study of attenuation correction algorithms in single photon emission computed tomography (SPECT), *Eur. J. Nucl. Med.* **13** (1987), 55-62

[Na] F.Natterer, *The Mathematics of Computerized Tomography* (Stuttgart: Teubner), 1986

[No] R.G.Novikov, An inversion formula for the attenuated X-ray transformation, *Ark. Mat.* **40** (2002), 145-167

[R] J.Radon, Uber die Bestimmung von Funktionen durch ihre Integralwerte langs gewisser Mannigfaltigkeiten, *Ber. Verh. Sachs. Akad. Wiss. Leipzig, Math-Nat.*, **K**1 **69** (1917), 262-267